MATH 135 Lecture Notes
December 12, 2006

Section 3.8
Differentiation of Implicit Functions

p. 106. An equation in which y is not expressed explicitly in terms of x may determine one or more functions. *Any such function, where y is defined implicitly as a function of x, is called an* **implicit function**.

A good example is the relation in x and y which defines the equation of a circle such as $x^2 + y^2 = 9$. We know that this relation is not a function since a circle fails the vertical line test. Furthermore, when we solve for y in terms of x we get two functions, $y = \pm\sqrt{9 - x^2}$.
We may enter these into the calculator as Y1= √(9 – X^2) and Y2=- √(9 – X^2).
This is an alternative method of graphing a circle. Previously we had used the parametric equations $x = 3\cos t$ and $y = 3\sin t$.

We know how to take the derivative of each of the two explicit functions. For Y1 the derivative is $\frac{dy}{dx} = \frac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{9 - x^2}}$. Implicit differentiation often allows us to find the derivative more easily. We will use implicit differentiation to find $\frac{dy}{dx}$ when $x^2 + y^2 = 9$.
First take the derivative with respect to x of each term and apply the chain rule:
$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(9)}{dx} \quad \Rightarrow 2x + \frac{d(y^2)}{dy}\frac{dy}{dx} = 0 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$. Check with the result above to see that they are equivalent.

Reality check: Find $\frac{dy}{dx}$ for each of these points on the graph of $x^2 + y^2 = 9$ and verify visually that the result is plausible: (0, 1), (1, 0), $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$, $\left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

Now let's find the second derivative for $x^2 + y^2 = 9$ and use it to locate where the circle is concave up and where it is concave down.

Apply the quotient rule to $\frac{dy}{dx} = -\frac{x}{y}$. $\quad \frac{d^2y}{dx^2} = -\frac{y \cdot 1 - x\frac{dy}{dx}}{y^2} = -\frac{y - x(-\frac{x}{y})}{y^2} = -\frac{y^2 + x^2}{y^3}$

Thus when $y > 0$, $\frac{d^2y}{dx^2} < 0$ and the graph is concave down. When $y < 0$, $\frac{d^2y}{dx^2} > 0$ and the graph is concave up.

Study these examples in the book:

Example 2, page 106. This is a conic section. Can you identify which type?

Example 3, page 107. Note use of the power rule.

Example 2, page 117, to find the equation of a tangent line.

Example 4, page 110 to find a second derivative. Note the use of the notation y' and y''.

Exercises page 108: 3, 5, 7, 9, 11, 21, 27, 29
Page 111: 25, 27 page 119: 8 and 12